

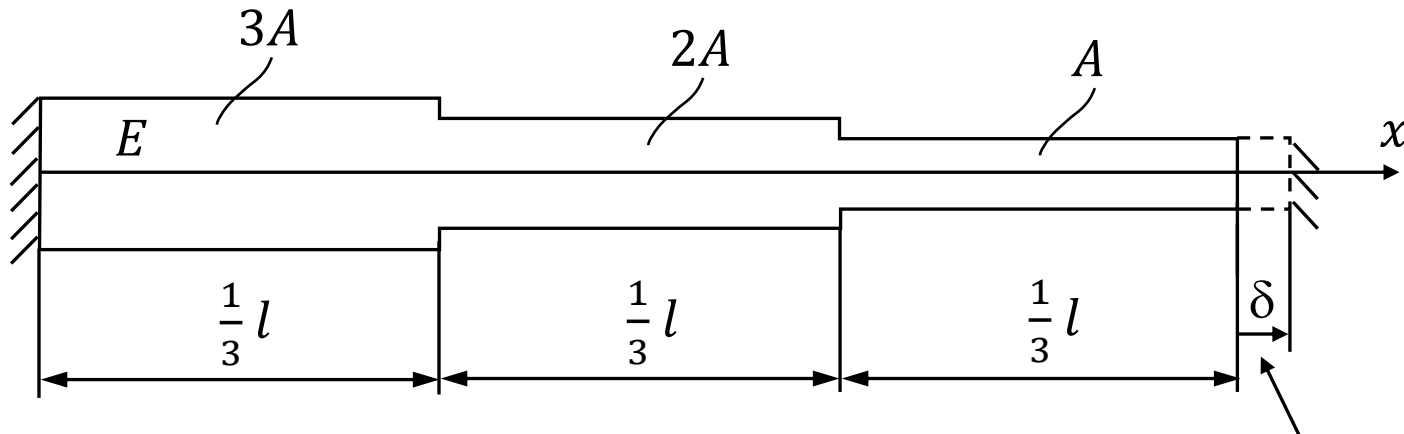


Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

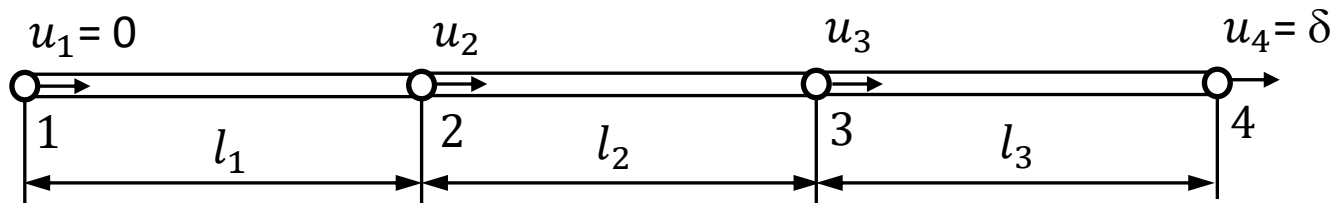
Imposed displacement and applied force

FE model of a bar with imposed displacement



imposed displacement

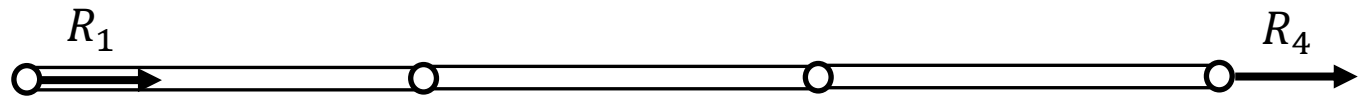
FE model – 3 bar elements:



$$[q] = [u_1, u_2, u_3, u_4]$$

1×4

FBD:



$$[F] = [R_1, 0, 0, R_4]$$

1×4

$(R_1 + R_4 = 0)$

FE model of a bar with imposed displacement

stiffness matrices:

$$[k]_e = \frac{EA_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k]_1 = \frac{E \cdot 3A}{\frac{1}{3}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{EA}{l} \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$[k]_2 = \frac{E \cdot 2A}{\frac{1}{3}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{EA}{l} \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}$$

$$[k]_3 = \frac{EA}{\frac{1}{3}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{EA}{l} \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$[k]_1^* = \frac{EA}{l} \begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; \quad [k]_2^* = \frac{EA}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 6 & -6 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k]_3^* = \frac{EA}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \rightarrow [K] = \sum_{e=1}^3 [k]_e^* = \frac{EA}{l} \begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 15 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

FE model of a bar with imposed displacement

set of equations:

$$[K] \cdot \{q\} = \{F\}$$

$4 \times 4 \quad 4 \times 1 \quad 4 \times 1$

$$[K] \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = [K] \cdot \left(\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{Bmatrix} \right) = [K] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} + [K] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{Bmatrix} = \{F\}$$

$$[K] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \{F\} - [K] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{Bmatrix} = \{F\} - \frac{EA}{l} \begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 15 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{Bmatrix} =$$

$$= \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ R_4 \end{Bmatrix} - \frac{EA}{l} \begin{Bmatrix} 0 \\ 0 \\ -3\delta \\ 3\delta \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ \frac{3EA}{l}\delta \\ R_4 - \frac{3EA}{l}\delta \end{Bmatrix} \rightarrow [K] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ \frac{3EA}{l}\delta \\ R_4 - \frac{3EA}{l}\delta \end{Bmatrix}$$

FE model of a bar with imposed displacement

boundary conditions: $u_1 = 0$

$$\frac{EA}{l} \begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 15 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ \frac{3EA}{l} \delta \\ R_4 - \frac{3EA}{l} \delta \end{Bmatrix}$$

u_2, u_3 - unknown nodal parameters

$$\frac{EA}{l} \begin{bmatrix} 15 & -6 \\ -6 & 9 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{3EA}{l} \delta \end{Bmatrix}$$

$$\begin{bmatrix} 15 & -6 \\ -6 & 9 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3\delta \end{Bmatrix}$$

$$\det \begin{bmatrix} 15 & -6 \\ -6 & 9 \end{bmatrix} = 15 \cdot 9 - (-6)(-6) = 99 \quad ; \quad \begin{bmatrix} 15 & -6 \\ -6 & 9 \end{bmatrix}^{CT} = \begin{bmatrix} 9 & 6 \\ 6 & 15 \end{bmatrix}$$

FE model of a bar with imposed displacement

displacements:

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{99} \begin{bmatrix} 9 & 6 \\ 6 & 15 \end{bmatrix} \begin{Bmatrix} 0 \\ 3\delta \end{Bmatrix}$$

$$u_2 = \frac{1}{99} (9 \cdot 0 + 6 \cdot 3\delta) = \frac{18\delta}{99} = \frac{2\delta}{11}$$

$$u_3 = \frac{1}{99} (6 \cdot 0 + 15 \cdot 3\delta) = \frac{45\delta}{99} = \frac{5\delta}{11}$$

$$N_1(\xi) = 1 - \frac{\xi}{l_e} = 1 - \frac{3\xi}{l}; \quad N_2(\xi) = \frac{\xi}{l_e} = \frac{3\xi}{l}$$

$$u(\xi) = [N_1, N_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e \rightarrow \varepsilon_x = \frac{du}{d\xi} = \left[\frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \left[-\frac{3}{l}, \frac{3}{l} \right] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_1 = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix}_1; \quad \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_2 = \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_2; \quad \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_3 = \begin{Bmatrix} u_3 \\ \delta \end{Bmatrix}_3$$

FE model of a bar with imposed displacement

strain in elements:

$$\varepsilon_{x1} = \left[-\frac{3}{l}, \frac{3}{l} \right] \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix}_1 = -\frac{3}{l} \cdot 0 + \frac{3}{l} \cdot u_2 = \frac{6\delta}{11l}$$

$$\varepsilon_{x2} = \left[-\frac{3}{l}, \frac{3}{l} \right] \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_2 = -\frac{3}{l} \cdot u_2 + \frac{3}{l} \cdot u_3 = \frac{9\delta}{11l}$$

$$\varepsilon_{x3} = \left[-\frac{3}{l}, \frac{3}{l} \right] \begin{Bmatrix} u_3 \\ \delta \end{Bmatrix}_3 = -\frac{3}{l} \cdot u_3 + \frac{3}{l} \cdot \delta = \frac{18\delta}{11l}$$

stress in elements:

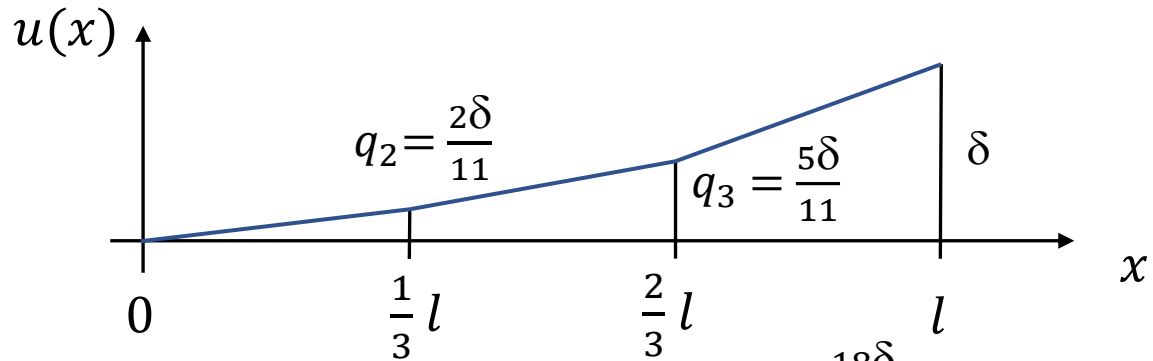
$$\sigma_{x1} = E \cdot \varepsilon_{x1} = \frac{6E\delta}{11l}$$

$$\sigma_{x2} = E \cdot \varepsilon_{x2} = \frac{9E\delta}{11l}$$

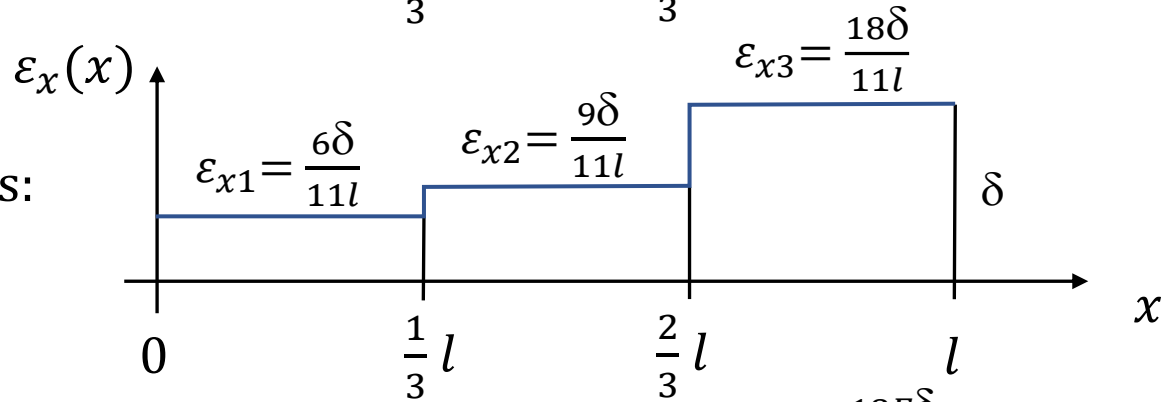
$$\sigma_{x3} = E \cdot \varepsilon_{x3} = \frac{18E\delta}{11l}$$

FE model of a bar with imposed displacement

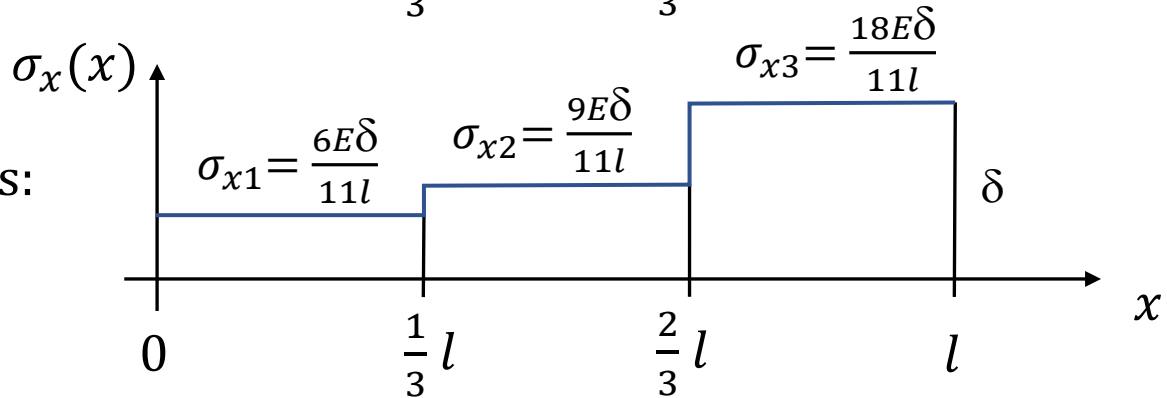
DOF solution:



Strain in elements:

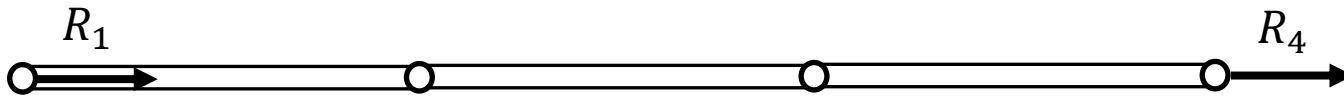


Stress in elements:



FE model of a bar with imposed displacement

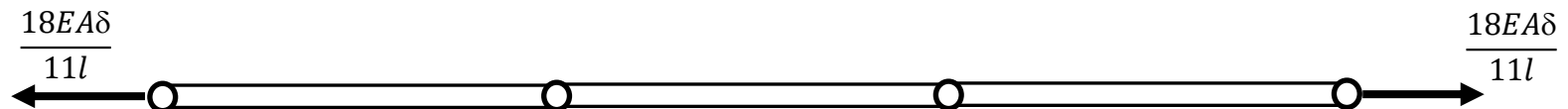
reactions:



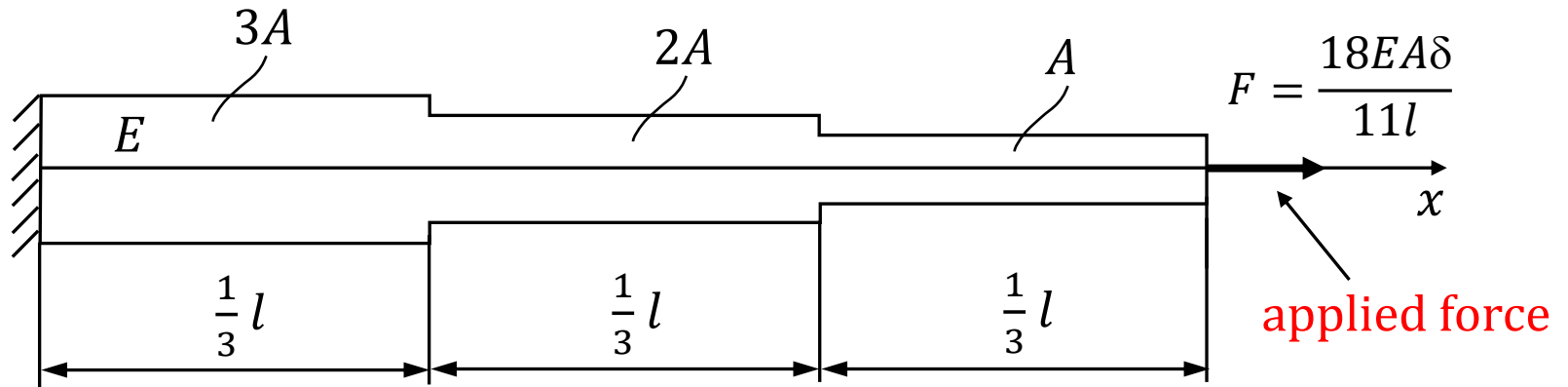
$$\frac{EA}{l} \begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 15 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{2\delta}{11} \\ \frac{5\delta}{11} \\ \delta \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ R_4 \end{Bmatrix}$$

$$\frac{EA}{l} \left(9 \cdot 0 - 9 \cdot \frac{2\delta}{11} + 0 \cdot \frac{5\delta}{11} + 0 \cdot \delta \right) = R_1 \quad \rightarrow \quad R_1 = -\frac{18EA\delta}{11l}$$

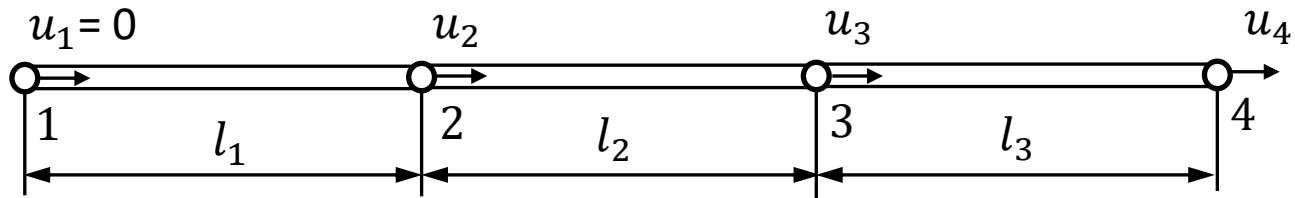
$$\frac{EA}{l} \left(0 \cdot 0 + 0 \cdot \frac{2\delta}{11} - 3 \cdot \frac{5\delta}{11} + 3 \cdot \delta \right) = R_4 \quad \rightarrow \quad R_4 = \frac{18EA\delta}{11l}$$



FE model of a bar with applied force



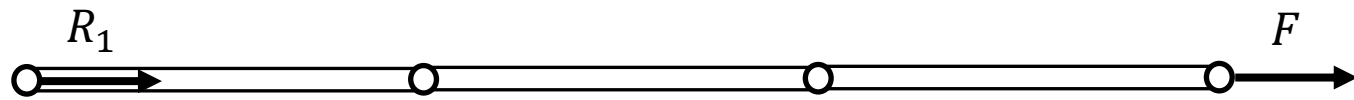
FE model – 3 bar elements:



$$[q] = [u_1, u_2, u_3, u_4]$$

1×4

FBD:



$$[F] = [R_1, 0, 0, F]$$

1×4

$$(R_1 + F = 0)$$

FE model of a bar with applied force

boundary conditions: $u_1 = 0$

$$\frac{EA}{l} \begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 15 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ F \end{Bmatrix}$$

u_2, u_3, u_4 - unknown nodal parameters

$$\frac{EA}{l} \begin{bmatrix} 15 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{18EA\delta}{11l} \end{Bmatrix}$$

$$\begin{bmatrix} 15 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{18\delta}{11} \end{Bmatrix}$$

FE model of a bar with applied force

$$\det \begin{bmatrix} 15 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix} =$$

$$= 15 \cdot (9 \cdot 3 - (-3)(-3)) - (-6) \cdot ((-6) \cdot 3 - (-3) \cdot 0) = 162$$

$$\begin{bmatrix} 15 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix}^{CT} = \begin{bmatrix} 18 & 18 & 18 \\ 18 & 45 & 45 \\ 18 & 45 & 99 \end{bmatrix}$$

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{1}{162} \begin{bmatrix} 18 & 18 & 18 \\ 18 & 45 & 45 \\ 18 & 45 & 99 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \frac{18\delta}{11} \end{Bmatrix}$$

FE model of a bar with applied force

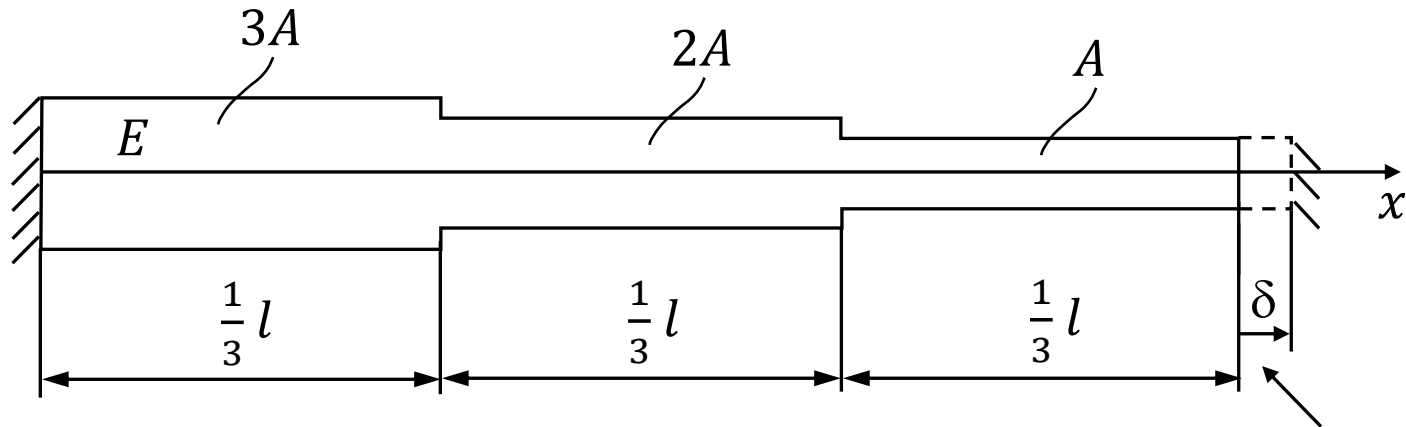
$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{1}{162} \begin{bmatrix} 18 & 18 & 18 \\ 18 & 45 & 45 \\ 18 & 45 & 99 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \frac{18\delta}{11} \end{Bmatrix}$$

$$u_2 = \frac{1}{162} \left(18 \cdot 0 + 18 \cdot 0 + 18 \cdot \frac{18\delta}{11} \right) = \frac{18 \cdot 18 \delta}{162 \cdot 11} = \frac{18 \cdot 9 \cdot 2 \delta}{18 \cdot 9 \cdot 11} = \frac{2\delta}{11}$$

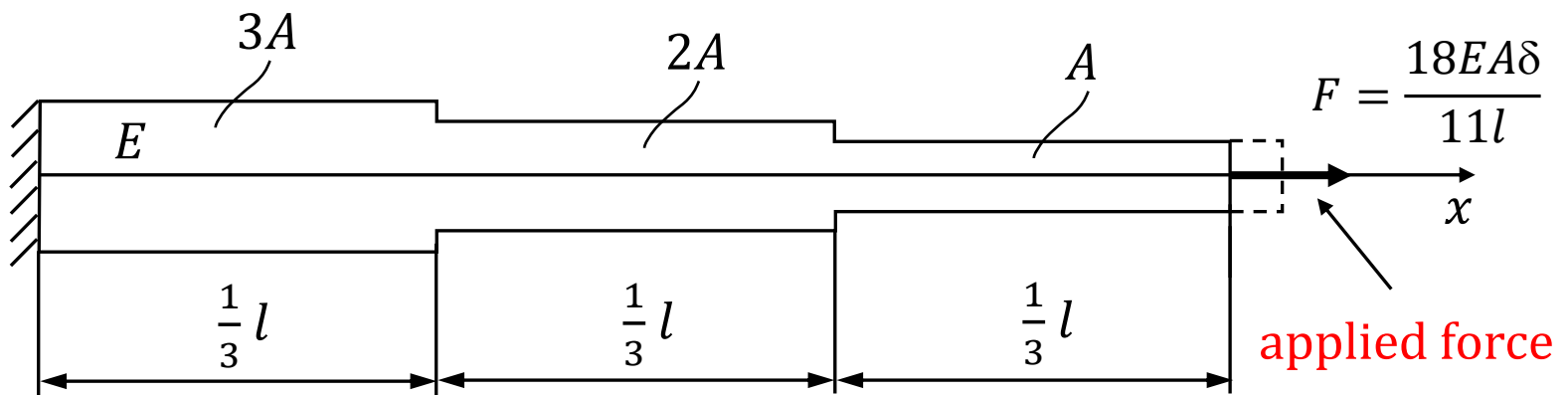
$$u_3 = \frac{1}{162} \left(18 \cdot 0 + 45 \cdot 0 + 45 \cdot \frac{18\delta}{11} \right) = \frac{45 \cdot 18 \delta}{162 \cdot 11} = \frac{5 \cdot 9 \cdot 18 \delta}{18 \cdot 9 \cdot 11} = \frac{5\delta}{11}$$

$$u_4 = \frac{1}{162} \left(18 \cdot 0 + 45 \cdot 0 + 99 \cdot \frac{18\delta}{11} \right) = \frac{99 \cdot 18 \delta}{162 \cdot 11} = \frac{11 \cdot 9 \cdot 18 \delta}{18 \cdot 9 \cdot 11} = \frac{11\delta}{11} = \delta$$

Two types of load giving the same result:



imposed displacement



Comparison between imposed displacement and force

Load type	Set of FE equations	N
Imposed displacement δ	$\frac{EA}{l} \begin{bmatrix} 15 & -6 \\ -6 & 9 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{3EA}{l} \delta \end{Bmatrix}$ $cond \left(\begin{bmatrix} 15 & -6 \\ -6 & 9 \end{bmatrix} \right) = 3.82$	2
Force $F = \frac{18EA\delta}{11l}$	$\frac{EA}{l} \begin{bmatrix} 15 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{18EA\delta}{11l} \end{Bmatrix}$ $cond \left(\begin{bmatrix} 15 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix} \right) = 16.4$	3

Condition number: $cond([K]) \approx 1$ - problem well conditioned, $(cond) \gg 1$ - ill conditioned

$$cond([K]) = \|K\|_{\infty} \cdot \|K^{-1}\|_{\infty}$$